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USING ASSUMED ENHANCED STRAIN ELEMENTS FOR LARGE COMPRESSIVE DEFORMATION

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Abstract—The formulation of the assumed enhanced strain element proposed by Simo and Armero [(1992). Geometrically non-linear enhanced strain mixed methods and the method of incompatible modes. *Int. J. Num. Methods Engng* **33**, 1413–1449] encountered difficulties for large compressive deformation with hyperelastic or elastic/plastic materials. The improved formulation presented by Simo *et al.* [(1993). Improved versions of assumed enhanced strain tri-linear elements for finite deformation problems. *Comp. Methods Appl. Mech. Engng* **110**, 359–386] proposes to alleviate this deficiency with a modified quadrature rule and shape function derivative calculation. In this work, we show an alternative approach which attempts to avoid this limitation by treating the orthogonality constraint on the enhanced field in rate form. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Practical considerations, such as automatic mesh generation, rezoning, enforcing contact constraints, and computational speed, promote the use of low-order elements. It is well-known, however, that the standard bi-linear quadrilateral and the tri-linear brick elements perform poorly in bending dominated problems and show a "locking" response in the nearly incompressible limit. Reduced integration elements provide relief at the expense of accuracy and the introduction of spurious zero-energy modes, requiring stabilization. It is not surprising, then, that the quest for an all-purpose, locking-free, fully-integrated, first-order element that can be used for nonlinear applications has received considerable attention in the literature.

Mixed finite element techniques have made significant inroads toward overcoming some of these limitations. As examples, for plasticity, the mean dilatation approach of Nagtegaal *et al.* (1974) circumvents locking in the incompressible limit. For plane stress elasticity, the interpolation procedure by Pian and Sumihara (1984) appears to be optimal in capturing inextensional states of bending stress. However, mixed methods (which take stress variables as additional solution quantities) suffer from the inherent inability to express the formulation in the strain-driven format required by the local integration algorithms for general inelastic materials.

For the infinitesimal theory, the classical method of incompatible modes in Wilson *et al.* (1973) provides a way to define a strain-driven formulation that performs well in bending dominated problems and the nearly incompressible limit. Within this context, Simo and Rifai (1990) provide a general strategy for constructing assumed strain finite element methods as a mixed method which incorporates the classical method of incompatible modes as a specific example. The extension of assumed strain or incompatible mode methods to nonlinear applications is not straightforward. In Simo and Armero (1992), a general methodology for performing this extension to the nonlinear theory is given. In addition to excellent performance in bending dominated states of stress and for nearly incompressible materials, the elements work well in capturing strain localization effects. The elements have been shown, however, to break down in highly constrained problems with large compressive deformation for both hyperelastic and elastic/plastic materials.

An improvement to the original formulation of the assumed enhanced strain elements is given by Simo *et al.* (1993). There, a modification to the integration rule, an approximate spatial gradient on the "hour-glass terms", and the inclusion of an additional volumetric mode appear to alleviate these deficiencies. They show very encouraging results for elastic/plastic materials under high compression.

We provide an alternative strategy for overcoming these limitations in the following sections. Our approach involves enforcing the orthogonality condition on the enhanced field through a linearized constraint written in rate form. This approach has shown considerable success.

In both the improvement by Simo *et al.* (total form) and the rate or incremental form outlined later, there still appears to be an important issue remaining. The enhanced assumed strain method is not (technically) an incompatible method, as the enhanced fields are not the gradient of an incompatible displacement field and continuity across element boundaries is not required. However, by adding an enrichment to the deformation gradient, the definition of the current volume of the element becomes ambiguous. For the method to be truly compatible, it would seem logical to require that the element volume measured by the position mapping (i.e., by the unenhanced deformation gradient) and the deformation gradient including the enhancement be the same. Otherwise, "fictitious" volume generation or loss can occur, disrupting mass continuity and energy balance. This volume equivalence motivates our statement of the central orthogonality condition in rate form. It does not appear possible to construct an enhancement that does not alter the volume at all, however, the rate of volume change can be minimized.

2. REVIEW OF THE FORMULATION

In this short work, we will not provide a full derivation or presentation of the enhanced strain formulation. For more complete details, see Simo and Armero (1992) and Simo *et al.* (1993), since these works form the basis for the discussion below. For clarity and definition of notation used in this presentation, we simply summarize the finite element weak form of the initial boundary value problem, restricted to geometrically nonlinear static analysis and hyperelastic constitutive relations.

Let $\Omega \subset R^{n_{dim}}$ ($n_{dim} = 2$ or 3 is the dimension of the space) be the reference configuration of the body with smooth boundary Γ. Points in the body are labeled $\mathbf{X} \in \Omega$. The deformation of the body is defined by the smooth map $\varphi(\cdot)$ on $\Omega \cup \Gamma$. Furthermore, let Γ_φ be the portion of the boundary in which the deformation is prescribed by $\bar{\varphi} = \varphi|_{\Gamma_\varphi}$. Points in the deformed body are labeled $\mathbf{x} \in \varphi(\Omega)$ and are given by

$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}), \quad \text{where } \mathbf{u}(\cdot) \text{ is the displacement field.}$$

The conforming part of the deformation gradient (the part defined by the deformation map) is

$$\frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathrm{GRAD_X}[\varphi] = \mathbf{1} + \mathrm{GRAD_X}[\mathbf{u}].$$

For clarity, we restrict the symbol $\mathbf{F}$ to be the sum of the conforming part of the deformation gradient plus the enhancement, as defined below.

The nominal stress tensor (unsymmetric, first Piola-Kirchhoff stress tensor) is denoted by $\mathbf{P}$ and the nominal traction $\mathbf{T} = \mathbf{PN}$ on the boundary Γ with normal $\mathbf{N}$ is specified on $\Gamma_T \subset \Gamma$ as $\bar{\mathbf{T}}$. The boundary partition satisfies $\Gamma_T \cap \Gamma_\varphi = \varnothing$ and $\overline{\Gamma_T \cup \Gamma_\varphi} = \Gamma$.

To state the weak form of the governing equations, we introduce the space $\mathscr{V}$ of conforming test functions:

$$\mathscr{V} = \{\delta\varphi(\cdot) : \delta\varphi(\mathbf{X}) = \mathbf{0} \quad \text{for} \quad \mathbf{X} \in \Gamma_\varphi\}.$$

Finite element approximations to all quantities are denoted by appending the superscript h, so that $\mathscr{V}^h \subset \mathscr{V}$ is the finite element approximation to the space of conforming test functions and φ^h is the finite element approximation to the deformation mapping φ, etc.

The details of the finite element approximations, however, are omitted since they are standard and not integral to the current discussion.

The key idea in the assumed enhanced strain method is that an independent field is added to the finite element approximation of the deformation gradient, $\mathbf{F}^h$, so that

$$\mathbf{F}^h = \mathrm{GRAD}_{\mathbf{X}}[\varphi^h] + \tilde{\mathbf{F}}^h,$$

where $\tilde{\mathbf{F}}^h \in \tilde{\mathscr{E}}^h$ is the enhanced field. Note that, as yet, the enhanced space $\tilde{\mathscr{E}}^h$ is not specified. In a standard finite element method, $\mathrm{GRAD}_{\mathbf{X}}[\varphi^h]$, referred to here as the conforming part, is the finite element approximation to the deformation gradient.

Let $\mathbf{B}$ be the body force per unit reference volume. Denote the L^2-inner product on Ω by $\langle \cdot, \cdot \rangle$ and the L^2-inner product on the boundary Γ by $\langle \cdot, \cdot \rangle_\Gamma$. The three-field Hu-Washizu functional in the variables $\langle \varphi, \tilde{\mathbf{F}}, \mathbf{P} \rangle$ in Simo and Armero (1992) can be written as two variational equations (see Simo *et al.* (1993)) which define the equilibrium equations.

$$\begin{aligned} \langle \mathbf{P}^h, \mathrm{GRAD}_{\mathbf{X}}[\delta\varphi^h] \rangle - \langle \mathbf{B}, \delta\varphi^h \rangle - \langle \mathbf{T}, \delta\varphi^h \rangle_\Gamma = 0 \quad & \forall \delta\varphi^h \in \mathscr{V}^h, \\ \langle \mathbf{P}^h, \delta\tilde{\mathbf{F}}^h \rangle = 0 \quad & \forall \delta\tilde{\mathbf{F}}^h \in T\tilde{\mathscr{E}}^h. \end{aligned} \tag{1}$$

Here, $T\tilde{\mathscr{E}}^h$ is the tangent space to the space of enhancements $\tilde{\mathscr{E}}^h$. In Simo and Armero (1992), $\tilde{\mathscr{E}}^h \subset \tilde{\mathscr{E}}$, where $\tilde{\mathscr{E}}$ is the linear space $\{\tilde{\mathbf{F}} : \Omega \to \mathscr{L}^{n_{dim}} : \tilde{F}^i_A \in L^2(\Omega)\}$, where $\mathscr{L}^{n_{dim}}$ is the vector space of real $n_{dim} \times n_{dim}$ matrices. In this case, $T\tilde{\mathscr{E}}^h = \tilde{\mathscr{E}}^h$. Since, for now, the space $\tilde{\mathscr{E}}^h$ is left unspecified, the tangent space $T\tilde{\mathscr{E}}^h$ is also undefined.

The nominal stress field $\mathbf{P}^h$ is computed from the constitutive equation with the enhanced deformation gradient as

$$\mathbf{P}^h = 2\mathbf{F}^h \, \partial_{\mathbf{C}} W(\mathbf{C}^h) \quad \text{where} \quad \mathbf{C}^h = \mathbf{F}^{h^T}\mathbf{F}^h. \tag{2}$$

Equations (1, 2), along with the boundary condition $\varphi^h|_{\Gamma_\varphi} = \bar{\varphi}^h$, define the boundary value problem for the finite element method implementation. With the exception of the specification of $\tilde{\mathscr{E}}^h$ (and $T\tilde{\mathscr{E}}^h$), the above formulation is as given in Simo and Armero (1992).

3. KINEMATIC REQUIREMENT ON THE ENHANCED FIELD

The conceptual goal of the enhanced method is clear. A solution dependent field, with an assumed form, is added to the deformation gradient such that the enriched deformation gradient leads to a finite element method that is free from the unenhanced method's poor performance (locking due to parasitic shear and Poisson's ratio effect in bending), while retaining most or all of its positive attributes.

The enhancement to the deformation gradient can not be arbitrary. In Simo and Armero (1992), two fundamental restrictions are placed on admissible fields:

(i) The tangent space to the enhanced field is L^2-orthogonal to constant nominal stress fields. From eqn $(1)_2$, this condition requires that

$$\int_V \delta\tilde{\mathbf{F}}^h \, \mathrm{d}V = \mathbf{0},$$

where V is the reference element volume.

(ii) The finite element space spanned by the independent enhanced fields is not contained in the finite element space spanned by the gradient of the displacement variations. This condition requires that

$$\mathrm{GRAD_X}[\mathscr{V}^h] \cap \tilde{\mathscr{E}}^h \text{ is empty.}$$

These two conditions ensure that the element passes the nonlinear version of the patch test (exact representation of the stress due to the total deformation which is homogeneous).

The patch test is a condition placed by equilibrium on the finite element sub-spaces. We propose that a further kinematic restriction be placed on the enhancement, one that states that the enhanced field alone (that is, for a fixed position mapping) can not add or subtract volume.

To see how to enforce this condition, we follow the standard construction of the enhanced field test functions. Consider a one-parameter curve of deformation gradients about the current configuration. The deformation gradient to the current configuration is defined by the gradient of the position mapping, φ^h, and the (non-zero) enhanced field, $\tilde{\mathbf{F}}^h$. Here, the position mapping is held constant and the enhancement is allowed to vary. The one-parameter curve of deformation gradients is defined by the linear relationship

$$\mathbf{F}^h_\varepsilon = \mathbf{F}^h + \varepsilon\delta\tilde{\mathbf{F}}^h. \tag{3}$$

Note that $\mathbf{F}^h_\varepsilon|_{\varepsilon=0} = \mathbf{F}^h = \mathrm{GRAD_X}[\varphi^h] + \tilde{\mathbf{F}}^h$ and $(\mathrm{d}/\mathrm{d}\varepsilon)|_{\varepsilon=0}\tilde{\mathbf{F}}^h_\varepsilon = \delta\tilde{\mathbf{F}}^h$. It is important to note that for the current configuration, the enhanced field $\tilde{\mathbf{F}}^h$ is (possibly) non-zero.

The deformation gradient $\mathbf{F}^h$ is a linear, orientation preserving transformation between the reference and current configurations. Therefore, $\mathbf{F}^h_\varepsilon$ in equation (3) must be a linear, orientation preserving transformation between the reference and current configurations. This requirement only restricts the admissible fields $\delta\tilde{\mathbf{F}}^h$ to be linear transformations between the reference and current configurations which preserve the condition $\det[\mathbf{F}^h_\varepsilon] > 0$.

It is our concern that this freedom to choose the test functions $\delta\tilde{\mathbf{F}}^h$ as any linear transformation gives too much freedom to the enhancements to alter the volume measured by the deformation gradient. We feel that for ε small, the enhanced field alone can not change the measured volume of the body. This is equivalent to the statement that the instantaneous rate of change of the volume due to enhanced field is zero. This requirement places the following restriction on the test functions $\delta\tilde{\mathbf{F}}^h$:

$$\int_v \mathrm{tr}\,(\delta\tilde{\mathbf{F}}^h\mathbf{F}^{h^{-1}})\,\mathrm{d}v = 0, \tag{4}$$

where the current element volume measure is $\mathrm{d}v = \det\mathbf{F}^h j_0(\xi)\,\mathrm{d}\square$, $j_0(\xi)$ is the Jacobian determinant of the reference configuration, $\xi \in \square$, and $\square$ is the isoparametric domain.

The condition (4) is derived as follows. The current element volume is defined

$$\text{volume} = \int_\square \det\mathbf{F}^h j_0(\xi)\,\mathrm{d}\square.$$

By a standard identity for the determinant function,

$$\det\mathbf{F}^h_\varepsilon = \det\mathbf{F}^h + \mathrm{tr}(\varepsilon\delta\tilde{\mathbf{F}}^h\mathbf{F}^{h^{-1}})\det\mathbf{F}^h + o(\varepsilon\delta\tilde{\mathbf{F}}^h).$$

Note that for $n_{dim} = 2$, $\det\mathbf{F}^h_\varepsilon = \det\mathbf{F}^h + \mathrm{tr}(\varepsilon\delta\tilde{\mathbf{F}}^h\mathbf{F}^{h^{-1}})\det\mathbf{F}^h$. The perturbed volume is

$$\begin{aligned}\text{volume}_\varepsilon &= \int_\square \det\mathbf{F}^h_\varepsilon j_0(\xi)\,\mathrm{d}\,\square \\ &= \int_\square \{\det\mathbf{F}^h + \mathrm{tr}\,(\varepsilon\delta\tilde{\mathbf{F}}^h\mathbf{F}^{h^{-1}})\det\mathbf{F}^h + o(\varepsilon\delta\tilde{\mathbf{F}}^h)\}\,j_0(\xi)\,\mathrm{d}\,\square\end{aligned}$$

$$= \text{volume} + \int_{\square} \{\operatorname{tr}(\varepsilon\delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}}) \det \mathbf{F}^h + o(\varepsilon\delta\tilde{\mathbf{F}}^h)\} j_0(\xi)\, \mathrm{d}\,\square.$$

For ε small (or in the limit as the increment in time goes to zero), the $o(\varepsilon\delta\tilde{\mathbf{F}}^h)$ terms can be neglected. (For $n_{dim} = 2$, this term is zero.) Since we require that the volume remain unchanged, the condition (4) follows.

The condition (4) does not imply that the enhancement $\tilde{\mathbf{F}}^h$ does not contribute to the overall volume. In a nonlinear finite element solution procedure, it implies that the increment in the enhancement is orthogonal to the volume of the element at the beginning of the increment. As the position mapping φ^h changes and the enhancement $\tilde{\mathbf{F}}^h$ is accumulated, the element volume measured by $\mathbf{F}^h$ will be different from that measured by $\mathrm{GRAD}_{\mathbf{X}}[\varphi^h]$. Instantaneously, however, to first order, increments $\Delta\tilde{\mathbf{F}}^h$ in the enhanced field should not change the volume of the element if the displacement increment is zero.

This condition is a kinematic constraint derived from an assumption about the way the element volume is permitted to change due to a perturbation of the enhanced field. Returning to equilibrium, however, this condition can be interpreted as requiring that the variation in the enhancement is orthogonal to a piecewise constant pressure field, $p^h = -(1/n_{dim}) \operatorname{tr} \boldsymbol{\sigma}^h$, where $\boldsymbol{\sigma}^h$ is the true or Cauchy stress tensor. To see this, recall that the nominal stress is related to the Cauchy stress through

$$\mathbf{P}^h = \det \mathbf{F}^h \boldsymbol{\sigma}^h \mathbf{F}^{h^{-T}}.$$

Write $\boldsymbol{\sigma}^h$ as a deviatoric part $\mathbf{s}^h$ and the hydrostatic pressure, $\boldsymbol{\sigma}^h = \mathbf{s}^h - \mathbf{p}^h\mathbf{1}$. Then equation $(1)_2$ becomes

$$\begin{aligned} 0 &= \langle \mathbf{P}^h, \delta\tilde{\mathbf{F}}^h \rangle \\ &= \langle \det \mathbf{F}^h \boldsymbol{\sigma}^h, \delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}} \rangle \\ &= \langle \det \mathbf{F}^h \mathbf{s}^h, \delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}} \rangle - \langle \det \mathbf{F}^h \mathbf{p}^h \mathbf{1}, \delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}} \rangle \\ &= \langle \det \mathbf{F}^h \mathbf{s}^h, \delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}} \rangle - \int_{\Omega^h} p^h \operatorname{tr}\left(\delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}}\right) \det \mathbf{F}^h \,\mathrm{d}\Omega^h, \end{aligned}$$

where, for clarity, the definition of the L^2-inner product is written explicitly for the pressure term. If we require that the enhancement be orthogonal to piecewise constant pressure fields, then the last integral must vanish over each element. This yields the volume condition in eqn (4), one which the test functions $\delta\tilde{\mathbf{F}}^h$ must satisfy in each element:

$$0 = \int_v \operatorname{tr}(\delta\tilde{\mathbf{F}}^h \mathbf{F}^{h^{-1}})\, \mathrm{d}v.$$

We view this condition as a third requirement placed on the test functions $\delta\tilde{\mathbf{F}}^h$ in addition to conditions i and ii stated earlier.

The above volume condition places a nonlinear constraint on the enhanced field $\tilde{\mathbf{F}}^h$. Hence, the space of enhancements $\tilde{\mathscr{E}}^h$ is no longer a linear space. Constructing the enhancement $\tilde{\mathbf{F}}^h \in \tilde{\mathscr{E}}^h$ such that $\delta\tilde{\mathbf{F}}^h \in T\tilde{\mathscr{E}}^h$ satisfies eqn (4) does not appear to be straightforward.

4. INCREMENT IMPLEMENTATION IN ABAQUS

Satisfaction of the volume condition above and the incremental version of the nonlinear patch test motivate the ABAQUS incremental formulation. The nonlinear patch test requires that a patch of elements exactly represents a homogeneous (total) deformation field. Furthermore, if we require that a patch of elements exactly represents a homogeneous deformation superposed on the (non-trivially) deformed configuration, we get the incremental nonlinear patch test. The primary idea is that the enhanced field is constructed on the element geometry at the beginning of the increment in a nonlinear solution procedure and added to the incremental deformation gradient. For more complete details of the ABAQUS implementation, see the ABAQUS Theory Manual (1994). In the construction that follows, approximate satisfaction of the volume condition occurs as a byproduct of the satisfaction of the incremental nonlinear patch test.

The deformation gradient at any time during the deformation can be written as the product of a series of incremental deformations:

$$\mathbf{F}^h_{n+1} = \Delta\mathbf{F}^h_{n+1}\Delta\mathbf{F}^h_n \ldots \Delta\mathbf{F}^h_1.$$

Let i denote the increment that advances the solution from time t to time $t+\Delta t$; then we write

$$\Delta\mathbf{F}^h_i = \frac{\partial \mathbf{x}^h_{t+\Delta t}}{\partial \mathbf{x}^h_t} + \Delta\tilde{\mathbf{F}}^h.$$

Denote the Jacobian transformation from the isoparametric domain to the beginning of the increment (time t) by $\mathbf{J}_t(\boldsymbol{\xi})$, so that

$$\mathbf{J}_t(\boldsymbol{\xi}) = \frac{\partial \mathbf{x}_t}{\partial \boldsymbol{\xi}} \quad \text{and} \quad j_t(\boldsymbol{\xi}) = \det(\mathbf{J}_t(\boldsymbol{\xi})).$$

We construct the enhancement to the incremental deformation gradient in the following way:

$$\Delta\tilde{\mathbf{F}}^h(\boldsymbol{\xi}) = \frac{j_t(\mathbf{0})}{j_t(\boldsymbol{\xi})}\Delta\tilde{\mathbf{g}}(\boldsymbol{\xi})\mathbf{J}_t(\mathbf{0})^{-1}.$$

The field $\Delta\tilde{\mathbf{g}}$ is constructed between the parametric domain and the deformed element domain at time t as

$$\Delta\tilde{\mathbf{g}}(\boldsymbol{\xi}) = \Delta\boldsymbol{\alpha}^i \otimes \boldsymbol{\xi}_i,$$

where $\Delta\boldsymbol{\alpha}^i (i = 1, \ldots, n_{dim})$ are independent degrees of freedom and $\boldsymbol{\xi}_i$ are the vectors, in two-dimensions,

$$\boldsymbol{\xi}_1 = \begin{Bmatrix} \xi_1 \\ 0 \end{Bmatrix} \quad \text{and} \quad \boldsymbol{\xi}_2 = \begin{Bmatrix} 0 \\ \xi_2 \end{Bmatrix},$$

or, in three-dimensions,

$$\boldsymbol{\xi}_1 = \begin{Bmatrix} \xi_1 \\ 0 \\ 0 \end{Bmatrix}, \quad \boldsymbol{\xi}_2 = \begin{Bmatrix} 0 \\ \xi_2 \\ 0 \end{Bmatrix} \quad \text{and} \quad \boldsymbol{\xi}_3 = \begin{Bmatrix} 0 \\ 0 \\ \xi_3 \end{Bmatrix}.$$

The enhancement to the incremental deformation gradient can be written in terms of the independent degrees of freedom as

$$\Delta\bar{\mathbf{F}}^h(\xi) = \Delta\boldsymbol{\alpha}^i \otimes \frac{j_t(0)}{j_t(\xi)}\mathbf{J}_t(\mathbf{0})^{-T}\xi_i.$$

To see that the above construction passes the incremental patch test, use the identity

$$\delta\mathbf{L} = \delta\mathbf{F}\mathbf{F}^{-1} = \delta(\Delta\mathbf{F})\Delta\mathbf{F}^{-1}.$$

Introducing the expressions for the deformation gradient in terms of the enhancements, we see that

$$\delta\mathbf{L}^h = \mathrm{GRAD}_{\mathbf{X}}[\delta\varphi^h]\mathbf{F}^{h^{-1}} + \delta\bar{\mathbf{F}}^h\mathbf{F}^{h^{-1}} = \frac{\partial\delta\varphi^h}{\partial\mathbf{x}_t}\Delta\mathbf{F}_i^{h-1} + \delta(\Delta\bar{\mathbf{F}}^h)\Delta\mathbf{F}_i^{h-1},$$

or

$$\delta\bar{\mathbf{F}}^h\mathbf{F}^{h^{-1}} = \delta(\Delta\bar{\mathbf{F}}^h)\Delta\mathbf{F}_i^{h-1}.$$

To satisfy the incremental patch test, this quantity must be orthogonal to a piece-wise constant Cauchy stress field (eqn $(1)_2$ written in terms of $\boldsymbol{\sigma}^h$), or equivalently, its integral over the element's volume must vanish:

$$\begin{aligned}\int_{v_t}\delta\bar{\mathbf{F}}^h\mathbf{F}^{h^{-1}}\,\mathrm{d}v_t &= \int_{v_t}\delta(\Delta\bar{\mathbf{F}}^h)\Delta\mathbf{F}^{h^{-1}}\,\mathrm{d}v_t \\ &= \int_{\square}\left(\delta\boldsymbol{\alpha}^i\otimes\frac{j_t(\mathbf{0})}{j_t(\xi)}\mathbf{J}_t(\mathbf{0})^{-T}\xi_i\right)\Delta\mathbf{F}^{h^{-1}}j_t(\xi)\,\mathrm{d}\square \\ &= j_t(\mathbf{0})\delta\boldsymbol{\alpha}^i\otimes\int_{\square}\Delta\mathbf{F}^{h^{-T}}\mathbf{J}_t(\mathbf{0})^{-T}\xi_i\,\mathrm{d}\square.\end{aligned}$$

For an incremental deformation that is homogeneous, $\Delta\mathbf{F}^h$ is constant over the element and can be taken outside the integral. Since $\int_{\square}\xi_i\mathrm{d}\square = \mathbf{0}$ by construction, it follows that

$$\int_{v_t}\delta\bar{\mathbf{F}}^h\mathbf{F}^{h^{-1}}\,\mathrm{d}v_t = \mathbf{0}. \tag{5}$$

The trace of eqn (5) satisfies the volume condition stated in eqn (4) only for incremental deformations which are homogeneous. For non-homogeneous deformations, there is no guarantee that the volume constraint is satisfied.

5. NUMERICAL EXAMPLES

The assumed enhanced strain elements perform well in bending dominated problems and in strain localization problems in the original formulation by Simo and Armero (1992) and ABAQUS's incremental formulation (for several examples of the ABAQUS implementation, see the ABAQUS/Standard Example Problems Manual). What is at issue is the element's performance in compression. We present below a representative simulation (the upsetting of a cylindrical billet) with an elastic/plastic material. The level of deformation is not extreme; the results shown are accurate. (Since this is a standard numerical test problem, accuracy is defined in comparison with other accepted simulations.) However, for this level of deformation, the original assumed enhanced strain formulation breaks down.

This example is an extension of the standard test case by Lippmann (1979). The specimen is a cylindrical billet, 30 mm long, with a radius of 10 mm. It is compressed between two flat, rigid dies with perfectly rough (i.e., no slip) surfaces. The finite element

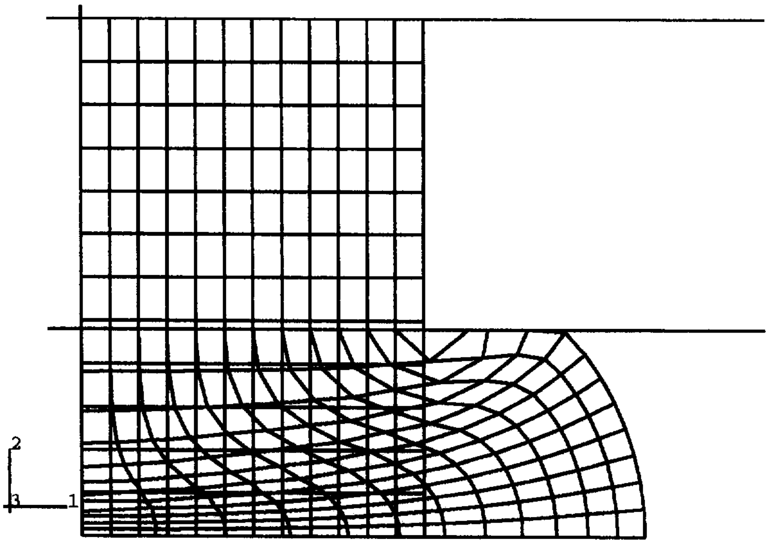

Fig. 1. Elastic/plastic compression of a cylindrical billet.

model consists of four-node axi-symmetric solid elements. Half symmetry of the axi-symmetric model is used, with a 12×12 mesh. The die displacement causes 60% upsetting. The material has Young's modulus 200 GPa, Poisson's ratio 0.3, initial yield stress 700 MPa, and work hardening rate 300 MPa.

6. CONCLUDING REMARKS

In this work, we argue for an additional constraint to be placed on the enhancement to the deformation gradient, one that requires that the variation of the enhanced field is orthogonal to a piece-wise constant pressure field. This condition has the kinematic interpretation that the instantaneous rate of change of the volume due to the enhanced field only (that is, with the nodal positions held fixed) is zero. We identified this as a deficiency in the original formulation of the assumed enhanced strain elements by Simo and Armero (1992). In the improvement by Simo *et al.* (1993), this condition remains unsatisfied. In the ABAQUS implementation, an incremental approximation to this volume constraint is used, whereby the enhanced field is constructed at the beginning of each increment and added to the incremental deformation gradient. Although this approach alleviates some of the original element's sensitivity to distortion under high compression, it does not appear to be a comprehensive solution.

We believe that satisfaction of the volume constraint is essential to obtain a robust formulation, particularly for applications in which large compressive strains and high hydrostatic pressures occur. Consequently, we are somewhat puzzled that Simo *et al.* (1993) are able to analyze problems of this type with a formulation that does not seem to satisfy the volume constraint. We hope that it will be possible to provide an explanation for this apparent contradiction.

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